THOUGHTS

Correspondences and Polynomials

Parameterized Families Let

$$k[x_i,t_j]\ni F_a(x_i,t_j) \text{ and } F\subseteq \mathbb{A}^m_x\times_k\mathbb{A}^n_t = X\times_k T=\operatorname{Spec}\big(k[x_i,t_j]\big)$$

be a system of polynomials and a closed subscheme $F=V(F_a)$ given by it. For the moment, the t_i are to be thought of as parameters.
We seek a decomposition

$$T=\sqcup T_\alpha$$

such that precisely on each T_α there is a uniformity of the fibers

$$F_t=V(F_a)_t$$

for $t\in T_\alpha$.
Ideally, we would like the T_α to be *semialgebraic sets* or *constructible sets.*

Properties There are enough properties according to which one could stratify:

1. F_t isomorphic to $F_{t'}$ for $t,t'\in T_\alpha$ in the sense of
   a) Isomorphism as k-varieties.
   b) Isomorphism as $\bar{k}$-varieties.
   c) Birational isomorphism.
   d) Isomorphism as analytic varieties.
   e) Isomorphism as topological spaces.
   f) Isomorphism as homotopy spaces.
2. F_t has exactly k preimages for $t\in T_k$ (possibly counted as the length of F_t).
3. F_t has exactly k irreducible components for $t\in T_k$.
4. F_t has dimension k for $t\in T_k$.
5. F_t is flat over T_α for $t\in T_\alpha$ (more precisely: $F_{T_\alpha}=V(F_a)_{T_\alpha}$ is flat over T_α).
6. Within $t\in T_\alpha$, F_t has “the same primary ideal types” (what does this mean at all? For $f(x_1,\dots,x_m,t)=0$, a hypersurface, it is clear; more generally one would first have to define it).
7. Within $t\in T_\alpha$, F_t has “the same singularities”. Here it is even less clear what this might mean at all. A zeroth-order approach would be: “Is nonsingular” vs. “Is singular”.
8. For $t,t'\in T_\alpha$, infinitesimally adjacent, F_t can always be deformed into $F_{t'}$.

Here one must always distinguish the cases of discrete moduli (finitely or countably many classes T_α) or continuous moduli (finitely many $G_a\in k[t_j,u_s]$ and finitely many $H_b(u_s)\in k[u_s]$, where the u_s are “implicit invariants” for the t_j).

Cubic Curves An example of continuous invariants would be, for instance,

$$F(x,y,a,b)=y^2-(x^3+ax+b)=0$$

as our function $F=0$, where x,y are the x_i and a,b are the t_j. The moduli space $k[t_j,u_s]$ is $k[a,b,j]$ with the function $G(a,b,j)=j(4a^3+27b^2)-1728\,4a^3$ (there are no functions $H_b(u_s)$ here).
However, one must still decompose the space $k[a,b]$ into $T_1=(4a^3+27b^2)\neq 0$ (elliptic curves) and $T_2=(4a^3+27b^2)=0, a\neq 0)$ (nodal curves) and $T_3=(4a^3+27b^2)=a=0)$ (cuspidal curves). The moduli space $(k[a,b,j],G)$ then only further subdivides the subspace T_1.
A hypothetical algorithm should find the decomposition of T into T_1,T_2,T_3 by itself and also the moduli space $k[a,b,j]$ for T_1 with the implicit invariant $G(a,b,j)$.

Kodaira-Spencer If, somewhat more generally, $V(F_a)=F\subseteq\mathbb{P}^m_{k[t_j]}$ is projective over T, thus the $F_a(x_i,t_j)$ are homogeneous in the x_i, then one has a Kodaira-Spencer map

$$K_t: T_tT\mapsto H^1(F_t,\mathcal{T}_{F_t})=T_{F_t}(\mathcal{M})$$

where $\mathcal{M}$ is the moduli space of the F_t.
For every algebraic invariant $u_s(t_j)$ which comes from $T\to\mathcal{M}$, one has

$$\ker du_s(t_j)|_t\supseteq\ker K_t.$$

The system of linear subspaces $\ker K_t$ defines a system of linear PDEs on (t_j) whose solutions are the algebraic invariants $u_s(t_j)$ that we sought above.

Kodaira-Spencer Example Let $T=\operatorname{Spec}(k[a,b])=\operatorname{Spec}(A)$ and $P=\mathbb{P}^2_T$ with $P=\operatorname{proj}(A[X,Y,Z])$, and further $X=V(F)\subseteq P$ with $F=F(a,b,X,Y,Z)$, for example

$$F=Y^2Z-(X^3+aXZ^2+bZ^3)$$

We have the sequence

$$0\to\pi^*\Omega_T\to\Omega_X\to\Omega_{X|T}\to 0$$

and the dualization $\mathcal{H}\mathrm{om}(-,\mathcal{O}_X)$

$$0\to\mathcal{T}_{X|T}\to\mathcal{T}_X\to\pi^*\mathcal{T}_T\to 0 \tag{1}$$

The diagram for the fiber at $t\in T$ is

$$\begin{tikzcd} X \arrow[d, "\pi"] & X_t \arrow[l, "j"'] \arrow[d, "\pi'"] \\ T & t \arrow[l, "i"'] \end{tikzcd}$$

Applying $j^*(-)$ to the sequence (1), we obtain

$$0\to\mathcal{T}_{X_t|t}\to\mathcal{T}_X\otimes_{\mathcal{O}_X}\mathcal{O}_{X_t}\to\mathcal{T}_{T,t}\otimes\mathcal{O}_{X_t}\to 0$$

The long exact sequence gives

$$H^0(\mathcal{T}_X\otimes_{\mathcal{O}_X}\mathcal{O}_{X_t})\xrightarrow{\psi_t}\mathcal{T}_{T,t}\otimes H^0(\mathcal{O}_{X_t})\xrightarrow{K_t}H^1(\mathcal{T}_{X_t|t}) \tag{2}$$

where K_t is the Kodaira-Spencer map and ψ gives a surjection onto $\ker K_t$.
Thus, if we want to determine $\ker K_t$ as $\operatorname{im}\psi_t$, we must describe $\mathcal{T}_X$ explicitly: At first it is

$$0\to\mathcal{T}_X\to\mathcal{T}_P\otimes_{\mathcal{O}_P}\mathcal{O}_X\to\operatorname{Hom}(\mathcal{I}/\mathcal{I}^2,\mathcal{O}_X) \tag{3}$$

where $\mathcal{I}$ is the ideal sheaf $F\mathcal{O}_P$.
Furthermore, $P=T\times_k\mathbb{P}^2_k$ and hence

$$\Omega_P=p_1^*\Omega_T\oplus p_2^*\Omega_{\mathbb{P}^2_k} \tag{4}$$

With the Euler sequence $0\to\Omega_{\mathbb{P}^2}\to\mathcal{O}_{\mathbb{P}^2}(-1)^3\to\mathcal{O}_{\mathbb{P}^2}\to 0$ and its dualization we obtain

$$\begin{aligned}\mathcal{T}_P&=p_1^*\mathcal{T}_T\oplus p_2^*\mathcal{T}_{\mathbb{P}^2_k}=\\&=\mathcal{O}_P\partial_a+\mathcal{O}_P\partial_b+(\mathcal{O}_P(1)\partial_X+\mathcal{O}_P(1)\partial_Y+\mathcal{O}_P(1)\partial_Z)/\mathcal{O}_P(X\partial_X+Y\partial_Y+Z\partial Z)\end{aligned} \tag{5}$$

With this representation by explicit vector fields, from the sequence (3) we obtain for $H^0(\mathcal{T}_X)$ the representation as the kernel of the map

$$v\in H^0(\mathcal{T}_P\otimes\mathcal{O}_X)\mapsto v\vdash F \tag{6}$$

and concretely we have

$$\begin{aligned}\mathcal{T}_P\otimes\mathcal{O}_X=(&\mathcal{O}_X\partial_a+\mathcal{O}_X\partial_b+\\&\mathcal{O}_X(1)\partial_X+\mathcal{O}_X(1)\partial_Y+\mathcal{O}_X(1)\partial_Z)/\mathcal{O}_X(X\partial_X+Y\partial_Y+Z\partial_Z)\end{aligned} \tag{7}$$

We can write this as

$$0\to\mathcal{L}\to\mathcal{W}\to\mathcal{T}_P\otimes\mathcal{O}_X\to 0 \tag{8}$$

and then consider the diagram from (3)

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] & H^0_{\geqslant 0}(\mathcal{W})/H^0_{\geqslant 0}(\mathcal{L}) \arrow[r, "v\vdash F"] & R(d) \\ 0 \arrow[r] & K' \arrow[r] \arrow[u] & H^0_{\geqslant 0}(\mathcal{W}) \arrow[r, "v\vdash F"] \arrow[u] & R(d) \arrow[u, "="] \end{tikzcd} \tag{9}$$

Then $\widetilde{K}=\mathcal{T}_X$ and $(K^{\text{sat}})_0=H^0(\widetilde{K})=H^0(\mathcal{T}_X)$.
The projections of K_0 and K_0' onto $A\partial_a+A\partial_b$ are identical, so that instead of determining K_0 we may also use the simpler K_0', where no quotient has to be taken.
From the sequence

$$0\to\mathcal{O}_P(-d+e)\xrightarrow{F}\mathcal{O}_P(e)\to\mathcal{O}_X(e)\to 0 \tag{10}$$

we obtain here, with $d=3$ and $e=0,1$, the equalities

$$H^0(\mathcal{O}_X)=H^0(\mathcal{O}_P)=A=k[a,b],\quad H^0(\mathcal{O}_X(1))=H^0(\mathcal{O}_P(1))=XA+YA+ZA \tag{11}$$

Thus the vector fields $v\in H^0(\mathcal{W})$ are always of the form

$$\begin{aligned}v=&A\partial_a+A\partial_b+\\&+(AX+AY+AZ)\partial_X+(AX+AY+AZ)\partial_Y+(AX+AY+AZ)\partial_Z\end{aligned} \tag{12}$$

or, written with functions,

$$\begin{aligned}v=&w_a(a,b)\partial_a+w_b(a,b)\partial_b+\\&+w_X(a,b;X,Y,Z)\partial_X+w_Y(a,b;X,Y,Z)\partial_Y+w_Z(a,b;X,Y,Z)\partial_Z\end{aligned} \tag{13}$$

where the w_X,w_Y,w_Z are linear and homogeneous of degree 1 in X,Y,Z.
These elements v which, when applied to F, annihilate it, i.e. give $v\vdash F=0$, are, projected onto $A\partial_a+A\partial_b=\mathcal{T}_T$, precisely the kernels of the Kodaira-Spencer maps K_t.
We thus obtain these w_a,w_b as parts of the solution of the system of linear PDEs

$$v\vdash F=0 \tag{14}$$

$$(w_a)_X,(w_a)_Y,(w_a)_Z=0 \tag{15}$$

$$(w_b)_X,(w_b)_Y,(w_b)_Z=0 \tag{16}$$

$$w_X=X\,w_X^X(a,b)+Y\,w_X^Y(a,b)+Z\,w_X^Z(a,b) \tag{17}$$

$$w_Y=X\,w_Y^X(a,b)+Y\,w_Y^Y(a,b)+Z\,w_Y^Z(a,b) \tag{18}$$

$$w_Z=X\,w_Z^X(a,b)+Y\,w_Z^Y(a,b)+Z\,w_Z^Z(a,b) \tag{19}$$

$$(w_X^X)_X,(w_X^X)_Y,(w_X^X)_Z=0 \tag{20}$$

$$\dots \tag{21}$$

$$(w_Z^Z)_X,(w_Z^Z)_Y,(w_Z^Z)_Z=0 \tag{22}$$

One can try, for example, to solve this system with a computer algebra system; the system *Maple* finds the desired solution in the example, equation (23).
Of course one does not really need to solve a system of LPDEs here; one simply takes all $w_a,w_b,w_X^X,\dots,w_Z^Z$ to be functions of a,b and computes in $v\vdash F=0$ all coefficients $c_{ijk}(a,b)$ of $X^iY^jZ^k$. They form a system of syzygy equations for the $w_a,w_b,w_X^X,\dots,w_Z^Z$, which can be solved by the usual Gröbner basis methods.
One obtains (in one way or another):

$$(w_a\partial_a+w_b\partial_b)G=2a\partial_aG+3b\partial_bG=0 \tag{23}$$

as the condition for an invariant $G=G(a,b)$. The general solution of this condition equation is

$$G(a,b)=g\Big(\frac{b}{a^{3/2}}\Big) \tag{24}$$

and indeed the known j-invariant $j=6912\frac{a^3}{4a^3+27b^2}$ can be expressed rationally in terms of the invariant b^2/a^3.

Kodaira–Spencer in General In general, $X\subseteq\mathbb{P}^n_T=P$ with $T=\operatorname{Spec}\big(k[t_j]\big)=\operatorname{Spec}(A)$ and $X=V(F_a(X_i,t_j))$ with $a=1,\dots,s$. Then

$$S=A[X_0,\dots,X_n],\quad I=(F_1,\dots,F_s),\quad R=S/I \tag{25}$$

and

$$P=\operatorname{proj}(S),\quad X=\operatorname{proj}(R). \tag{26}$$

For the sake of correctness, we assume that X is nonsingular, so that the sequence

$$0\to\mathcal{I}/\mathcal{I}^2\to\Omega_{P|k}\otimes\Omega_X\to\Omega_{X|k}\to 0 \tag{27}$$

is an exact sequence of sheaves.
We again use the sequence (1), which now forces us to compute the expression $H^0(\mathcal{T}_X)$ in (3).
We therefore write (3) as

$$0\to N\to\Big(\sum_j R\partial_{t_j}+\sum_i R(1)\partial_{X_i}\Big)/R\Big(\sum_i X_i\partial_{X_i}\Big)\xrightarrow{\delta}\operatorname{Hom}(I/I^2,R) \tag{28}$$

and note that, upon applying $\widetilde{(-)}$, the sheafification, one recovers the sequence (3). In particular, for

$$D_P = (\sum_j R\partial_{t_j} + \sum_i R(1)\partial_{X_i})/R(\sum_i X_i\partial_{X_i})$$

one always has $\widetilde{D}_P = \mathscr{T}_P \otimes \mathscr{O}_X$.
Now $\phi : M = \sum_\nu R(-d_\nu)e_\nu \to I/I^2$ with $\phi(e_\nu) = F_\nu + I^2$ and $d_\nu = \deg_X F_\nu$ is a surjection, so that we can also rewrite (28) as

$$0 \to N \to (\sum_j R\partial_{t_j} + \sum_i R(1)\partial_{X_i})/R(\sum_i X_i\partial_{X_i}) + \xrightarrow{\delta'} \operatorname{Hom}(M, R) \tag{29}$$

where

$$\delta'(\partial_{t_j}) \vdash \phi(e_\nu) = \partial_{t_j} F_\nu \tag{30}$$

$$\delta'(\partial_{X_i}) \vdash \phi(e_\nu) = \partial_{X_i} F_\nu \tag{31}$$

hold.
Thus we have clarified all terms and maps in the sequence (29) and can, using the usual techniques provided by Macaulay2, apply H^0(sheaf(−)) to the module sequence (29) and obtain $H^0(T_X)$ explicitly as a submodule of

$$T_P = H^0(\widetilde{D}_P) \tag{32}$$

.
Subsequently, we simply project the generators of T_P that lie in the kernel of δ' onto $H^0(\sum_j R\partial_{t_j})$, just as we did in the special case of a single hypersurface.

Applications If one had an algorithm that produced such stratifications, one could decide many things:

1. Is $F_a(x_i, y_j) = 0$ the graph of a morphism from $X' \subseteq X$ to $Y' \subseteq Y$?
2. Is $F_a(x_i, y_j) = 0$ an isomorphism between $X' \subseteq X$ and $Y' \subseteq Y$?

In case 1, one checks whether X' lies in a flatness stratum, i.e. whether $F_{X'}$ is flat over X', and whether X' lies in the stratum of all zero-dimensional fibers of length 1.
In case 2, one applies the criteria from 1 also "backwards".
In this way one should then be able to see that $t^2 - u, t^3 - v$ does not induce an isomorphism between $\mathbb{A}^1_t$ and $V(u^3 - v^2) \subseteq \mathbb{A}^2_{u,v}$.

Further Dreams

A book with the title "From Symbolic Calculation to Gröbner Bases" in which the whole of algebraic geometry is reduced to polynomials $F(x_i, y_j, \dots, z_k) = 0$ ("symbolic calculation") and all algebraic "predicates" occur as results of algorithmic computations (at the core always Gröbner bases). Elimination term orders and quantifier elimination play a major role.

A vector bundle is a system (x_i, t_j) with $F_a(x_i) = 0$, a map of vector bundles a system (x_i, t_j, y_k, u_k) with $G_b(x_i, t_j, y_k, u_k) = 0$, which is linear in t_j and u_k. Projective schemes are represented by $F_a(x_i, y_j, z_k) = 0$, which are homogeneous in each group x_i, y_j, z_k.

Instead of completions, power-series expansions are used. How far might one get with all this? "It's a long long way to Tipperary...".